EPFL

3. Microfluidic channels and circuits

3.1 Poiseuille flow (pressure-drive flow)

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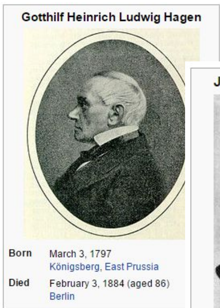
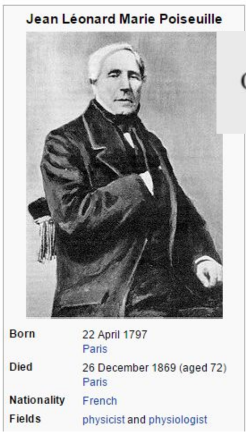
1839, **Hagen** introduced $\boldsymbol{Q \sim \Delta p \cdot d^4}$ in an older formula including also a Q^2 term.

$$\Delta p = \frac{1}{d^4}\left(aLQ + bQ^2\right)$$





$$Q = K'' \frac{d^4}{L} \Delta p$$

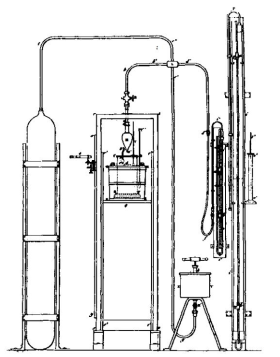


Figure 1 Frontal elevation view of Poiseuille's apparatus. Photocopy of one segment of a ten part fold-out plate published with Poiseuille's summary paper (1846).

Poiseuille's was interested in hemodynamics.

⇨ Extensive studies of liquid flow in small glass tubes (capillaries).

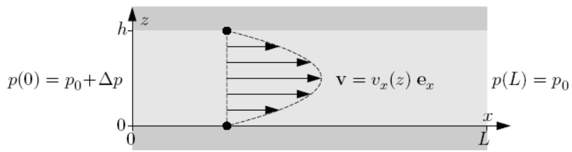
⇨ Only the first term ~$\boldsymbol{Q}$ was valid in this case.

Pressure-driven laminar flow between infinite parallel plates

EPFL

The flow is induced by a constant positive pressure difference Δp over a length L.

$$p(\mathbf{r}) = \frac{\Delta p}{L}(L - x) + p_0 \qquad (3.19)$$



(Fig. 3.6)

Navier-Stokes eqn.

$$\partial_z^2 v_x(z) = -\frac{\Delta p}{\eta L} \qquad v_x(0) = 0, \ \text{(no-slip)} \quad v_x(h) = 0, \ \text{(no-slip)} \qquad (3.28)$$

⇨ **Parabolic flow profile**

$$v_x(z) = \frac{\Delta p}{2\eta L}(h - z)z \qquad (3.29)$$

⇨ **Volumetric flow rate *Q***
(channel *w, h, L*)
[m³/s]

$$Q = \int_0^w \mathrm{d}y \int_0^h \mathrm{d}z\, \frac{\Delta p}{2\eta L}(h - z)z = \frac{h^3 w}{12\eta L}\Delta p \qquad (3.30)$$

Pressure-driven laminar flow between infinite parallel plates

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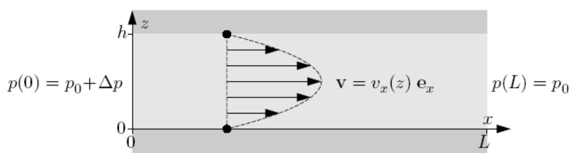


(Fig. 3.6)

⇨ **Volumetric flow rate *Q***
(channel *w, h, L*)
[m³/s]

$$Q = \int_0^w \mathrm{d}y \int_0^h \mathrm{d}z\, \frac{\Delta p}{2\eta L}(h - z)z = \frac{h^3 w}{12\eta L}\Delta p \qquad (3.30)$$

(3.30) is an approximation for real channels: error 23% for $w/h = 3$ and 7% for $w/h = 10$.

⇨ Suitable for wide and shallow channels with high aspect ratio $w/h >> 1$.

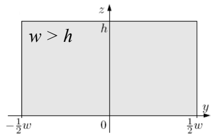
Poiseuille flow in channels with rectangular cross-section

EPFL

More details in *Henrik Bruus* “Theoretical Microfluidics”

$$\boxed{\left[\partial_y^2+\partial_z^2\right]v_x(y,z)=-\frac{\Delta p}{\eta L}} \quad \text{for} \quad -\frac{1}{2}w<y<\frac{1}{2}w,\ \ 0<z<h \tag{3.47}$$

$$v_x(y,z)=0, \qquad \text{for} \quad y=\pm\frac{1}{2}w,\ z=0,\ z=h.$$



⇨ Solution may be found by Fourier expansion (here in z):

$$v_x(y,z)\equiv\sum_{n=1}^{\infty}f_n(y)\sin\left(n\pi\frac{z}{h}\right) \quad \text{and} \quad -\frac{\Delta p}{\eta L}=-\frac{\Delta p}{\eta L}\,\frac{4}{\pi}\sum_{n,\text{odd}}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{z}{h}\right) \tag{3.48, 3.49}$$

in (3.47)

$$\left[\partial_y^2+\partial_z^2\right]v_x(y,z)=\sum_{n=1}^{\infty}\left[f_n''(y)-\frac{n^2\pi^2}{h^2}f_n(y)\right]\sin\left(n\pi\frac{z}{h}\right) \tag{3.50}$$

⇨ Equations for the coefficients have to be solved with boundary conditions.

$$\begin{aligned} f_n(y)&=0, && \text{for } n \text{ even},\\ f_n''(y)-\frac{n^2\pi^2}{h^2}f_n(y)&=-\frac{\Delta p}{\eta L}\,\frac{4}{\pi}\,\frac{1}{n}, && \text{for } n \text{ odd}. \end{aligned} \tag{3.51}$$

“Microfluidics” -- Thomas Lehnert -- EPFL (Lausanne)

1/15

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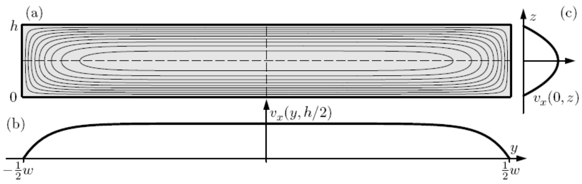
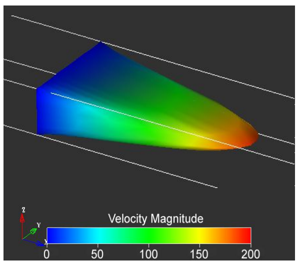
⇨ Velocity profile in a rectangular channel

$$v_x(y,z)=\frac{4h^2\Delta p}{\pi^3\eta L}\sum_{n,\text{odd}}^{\infty}\frac{1}{n^3}\left[1-\frac{\cosh\left(n\pi\frac{y}{h}\right)}{\cosh\left(n\pi\frac{w}{2h}\right)}\right]\sin\left(n\pi\frac{z}{h}\right)$$



(a): Contour lines for $v_x(y,z)$ in steps of 10% of $v_{x,max}$
(b) and (c): $v_x(y,h/2)$ and $v_x(0,z)$ along the respective centerlines.



Simulation of a pressure driven flow in a microchannel (aspect ratio 2:5). Velocity at the walls is zero.

⇨ Flow rate in a rectangular channel (approximation for $h/w \to 0$, $w > h$, flat and wide)

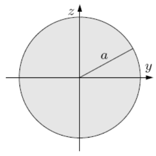
$$\boxed{Q=2\int_0^{\frac{1}{2}w}\mathrm{d}y\int_0^h\mathrm{d}z\ v_x(y,z)\ \approx\ \frac{h^3w\Delta p}{12\eta L}\left[1-0.630\,\frac{h}{w}\right]} \tag{3.58}$$

Good approximation ! Error 13% for $h = w$ and only 0.2% for $h = w/2$

Flow in capillaries with circular cross-section

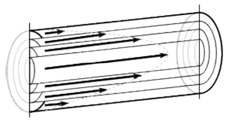
Navier-Stokes eqn. in cylindrical coordinates (*r, θ, x*) in a pipe with radius *a* (length *L*). A constant pressure drop *Δp* is applied along the *x*-direction. The flow profile is axisymmetric with respect to the center line (v_x is independent of *θ*). No-slip at the pipe wall (*r* = *a*).

$$\left[\partial_r^2 + \frac{1}{r}\,\partial_r\right] v_x(r) = -\frac{\Delta p}{\eta L} \qquad (3.39a)$$



…using a trial solution $v_x(r) = v_0\,(1 - r^2/a^2)$

⇨ Parabolic velocity profile $\boxed{v_x(r,\phi) = \frac{\Delta p}{4\eta L}\left(a^2 - r^2\right)}$



⇨ Flow rate Q for a small a tube $\boxed{Q = \frac{\pi a^4}{8\eta L}\,\Delta p}$ (3.42b)

Δp is the pressure difference
L is the length of pipe
η is the dynamic viscosity
Q is the volumetric flow rate
a is the pipe radius.

3.2 Hydraulic resistance and microfluidic networks

Poiseuille flow ⇨ *In all cases*: Linear relationship between *Q* and *Δp*

The proportional constant be defined as the **hydraulic resistance R_{hyd}** (or inverse conductance G^{-1}_{hyd}).

$$\boxed{\Delta p = R_{\mathrm{hyd}}\, Q = \frac{1}{G_{\mathrm{hyd}}}\, Q}$$

$$[Q] = \frac{\mathrm{m}^3}{\mathrm{s}} \qquad [\Delta p] = \mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{kg}}{\mathrm{m\,s}^2} \qquad [R_{\mathrm{hyd}}] = \frac{\mathrm{Pa\,s}}{\mathrm{m}^3} = \frac{\mathrm{kg}}{\mathrm{m}^4\,\mathrm{s}}$$

Hagen-Poiseuille law

⇨ Fluidic "Ohm's law" ($I_{el} \rightarrow Q$; $V_{el} \rightarrow \Delta p$; $R_{el} \rightarrow R_{hyd}$) for laminar flow conditions. It is of fundamental importance for the design of microfluidic circuits.

⇨ Further-going analogy:

Capacitance/Complianc $C_{el} \rightarrow C_{hyd}$

Inductance/Inertia $L_{hyd} = \rho L/A$

(relevant only for fluidic switching frequencies > 100 Hz and relative large/long channels (≥ mm-size).

⇨ The concept of impedance also applies, *e.g.* for an oscillating *p(t)* stimulus. **Equivalent electrical circuits** may be established to analyze the fluidic response of microfluidic circuits.

Hydraulic resistances for Poiseuille flow straight channels with different cross-sections

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shape		R_{hyd}	R_{hyd} $[10^{11}\ \frac{\mathrm{Pa\,s}}{\mathrm{m}^3}]$
circle	a	$\frac{8}{\pi}\,\eta L\,\frac{1}{a^4}$	0.25
two plates	h, w	$12\,\eta L\,\frac{1}{h^3 w}$	0.40
rectangle	h, w	$\frac{12\,\eta L}{1-0.63(h/w)}\,\frac{1}{h^3 w}$	0.51
square	h, h	$\frac{12\,\eta L}{1-0.917\times 0.63}\,\frac{1}{h^4}$	2.84
arbitrary		$\approx 2\eta L\,\mathcal{P}^2/\mathcal{A}^3$	P perimeter; A cross section

$$\boxed{\Delta p = R_{\mathrm{hyd}}\,Q = \frac{1}{G_{\mathrm{hyd}}}\,Q}$$

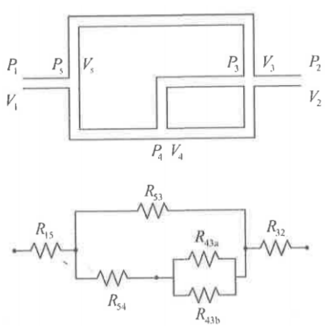
$$\boxed{Q = \frac{\pi a^4}{8\eta L}\,\Delta p}$$

Numerical values are for: water η = 1mPa·s, channel length L = 1 mm, a = 100 µm, b = 33 µm, h = 100 µm, w = 300 µm.

1/30 + application ?

Fluidic "Kirchhoff laws" at low Re-numbers

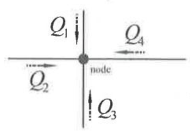
EPFL



Hydraulic circuit analog of a simple microfluidic device.

1) The total pressure drop Δp_i in a closed loop is zero.

2) The sum of flow rates Q_i at a node is zero (conservation of mass).



$$\Sigma Q = Q_1 + Q_2 + Q_3 + Q_4 = 0$$

R_{hyd} for a series connection of two channels ($\Delta p = \Sigma p_i$)

$$\boxed{R = R_1 + R_2} \quad (4.42)$$

R_{hyd} for a parallel connection of two channels ($Q = \Sigma Q_i$)

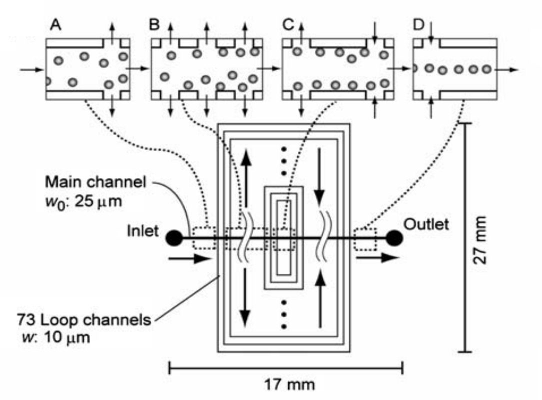
$$\boxed{R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}} \quad (4.43)$$

"Microfluidics" -- Thomas Lehnert -- EPFL (Lausanne)

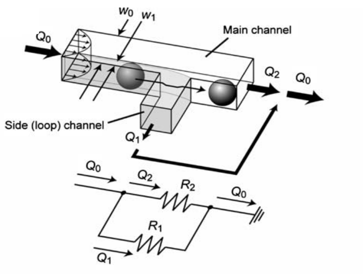
Example : Hydrodynamic focusing

R. Aoki et al., Microfluid Nanofluid, 6, 571-576 (2009)

The resistive circuit consists of the main channel and a loop channel.



Device with multiple loop microchannels: Repeated splitting and re-injection of flow in the main channel from both sides gradually focuses the particles in the center.

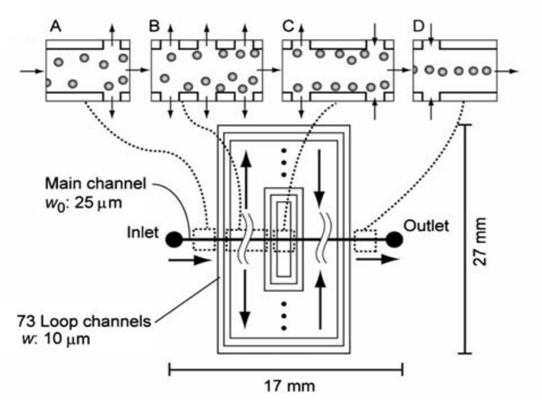


Branch point in the main channel. A small portion of the main flow section enters into the side channel (gray-colored).

Particles do not enter the loop channel. Diameter 3-5 µm > virtual width w_1 ~ µm.

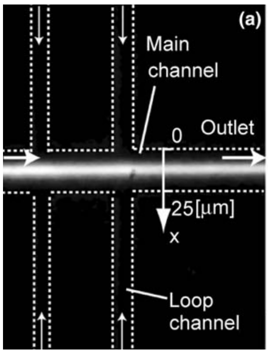
Example : Hydrodynamic focusing

R. Aoki et al., Microfluid Nanofluid, 6, 571-576 (2009)



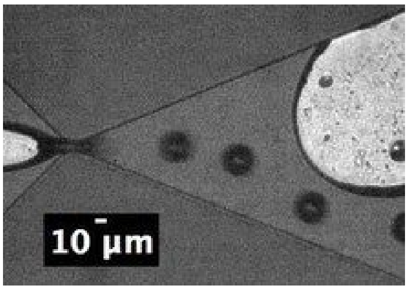
Device with multiple loop microchannels: Repeated splitting and re-injection of flow in the main channel from both sides gradually focuses the particles in the center.

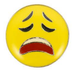
Particles (Ø = 5 µm) were focused within a width of a few µm around center line.



Focusing efficiency was not affected by the flow speed, here 100 mm/s, high velocity!

3.3 Compliance (hydraulic capacitance)





Air bubble in the system !

3.3 Compliance (hydraulic capacitance)

Compliance due to compressible gas (e.g. air bubble in microchannel)

$$\boxed{C_{\mathrm{hyd}} \equiv -\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}p}}$$

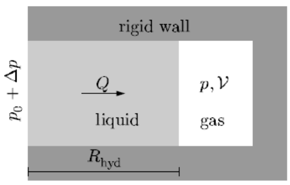
$C_{\mathrm{hyd}} = p_0\mathcal{V}_0/p^2$ *using* $p\mathcal{V} = p_0\mathcal{V}_0$

$$Q(t) = -\partial_t\mathcal{V} = -(\partial_p\mathcal{V})\partial_t p = C_{\mathrm{hyd}}\partial_t p$$

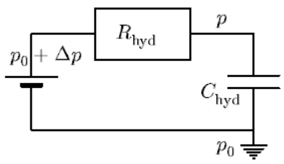
$$(p_0 + \Delta p) - p = R_{\mathrm{hyd}}Q = -R_{\mathrm{hyd}}\partial_t\mathcal{V} = R_{\mathrm{hyd}}C_{\mathrm{hyd}}\partial_t p$$

$$\boxed{p(t) = p_0 + \left(1 - e^{-t/\tau}\right)\Delta p, \quad \tau \equiv R_{\mathrm{hyd}}C_{\mathrm{hyd}}} \quad (4.47)$$

Analogy to voltage drop on charging capacitor



(Fig. 4.9)

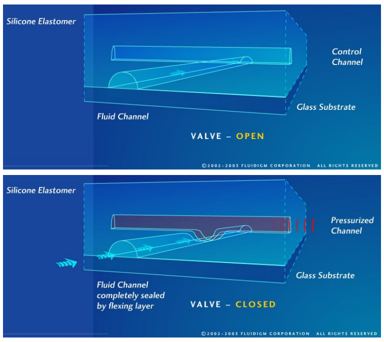


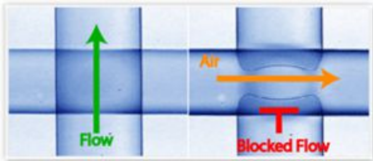
Equivalent electrical circuit

3.3 Compliance (hydraulic capacitance)

PDMS valves for pumps, mixers and large-scale integration

(see for example: *J. Melin and S. Quake, Annu. Rev. Biophys. Biomol*. Struct., 36:213–31, 2007)





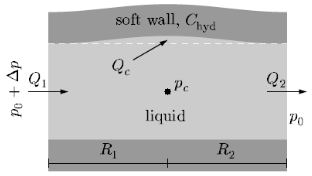
"Quake Valves" based on two crossing PDMS channels, *i.e.* a fluidic channel and a control channel, in open and closed state.

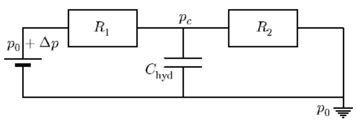
https://www.fluidigm.com/about/technology

3.3 Compliance (hydraulic capacitance)

PDMS valves for pumps, mixers and large-scale integration

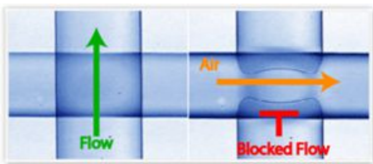
$$Q_c = -\partial_t \mathcal{V} = C_{\mathrm{hyd}} \partial_t p_c$$





Analog to voltage drop on capacitor charging through a voltage divider.

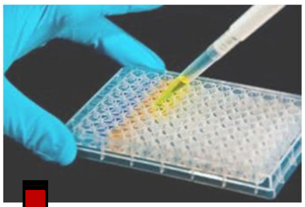
$$\boxed{p_c(t) = p_0 + \left(1 - e^{-\left[\tau_1^{-1} + \tau_2^{-1}\right]t}\right)\frac{\tau_2}{\tau_1 + \tau_2}\Delta p}$$

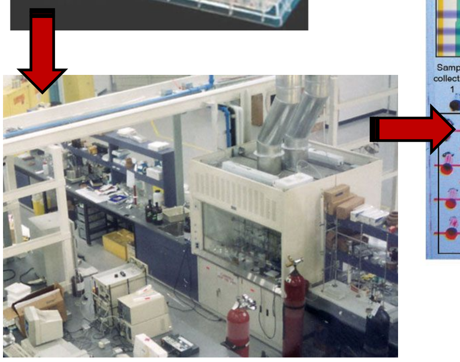


C_{hyd} (*i.e.* the membrane deformation) depends on the valve geometry and material parameters.

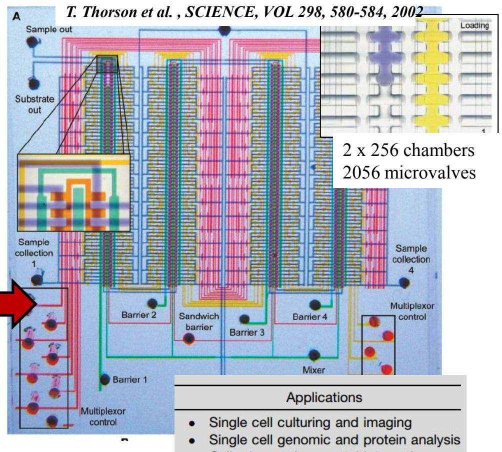
3.4 Microfluidic devices based on elastomeric components

Microfluidic large-scale integration for highly parallel and automated fluidic processes





T. Thorson et al. , SCIENCE, VOL 298, 580-584, 2002



2 x 256 chambers
2056 microvalves

Applications

- Single cell culturing and imaging
- Single cell genomic and protein analysis
- Cell microenvironmental interaction
- Protein synthesis and characterization
- Chemical and biological screening
- Solid phase liquid chromatography

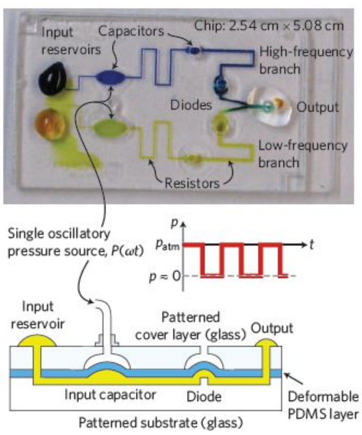
I.E. Araci and P. Brisk, *Recent developments in microfluidic large scale integration*, Current Opinion in Biotechnology 2013, 25:60–68

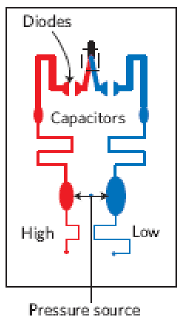
Example 2: RC-tuned flow control

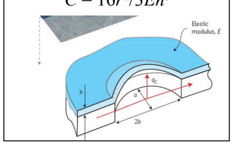
D. Leslie et al., Nature Physics 5, 231 - 235 (2009)

Two *RC* branches with diodes are connected in parallel and driven by a single oscillatory pressure source *p(t)*. Each branch (band-pass) has a different frequency response.

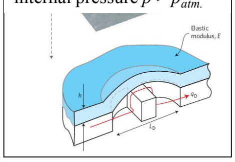
⇨ Changing the frequency switches the flow output from one channel to the other.







Compliance for circular shape
$C = 16r^6/3Eh^3$



Fluidic diode: opens if the internal pressure $p > p_{atm.}$

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D. Leslie et al., Nature Physics 5, 231 - 235 (2009)

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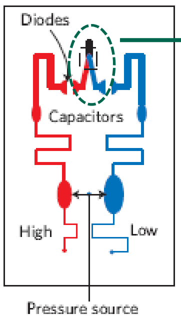
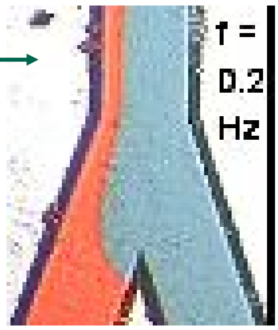
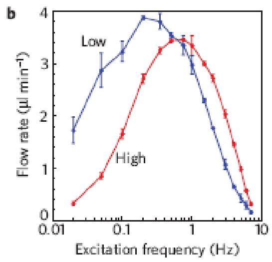




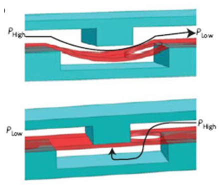
☹ large bandwidth, no “clean” switching !



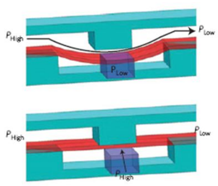
Band-pass characteristics of the time-averaged output contributions from each branch of the network.

Example 3: “Microfluidic electronics”

B. Mosadegh et al., Nature Physics, 6, 433-437 (2010)



fluidic diode



fluidic “field effect transistor”